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A rank criterion for the order of a pole of a matrix function

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Abstract

By treating the null chains of a meromorphic matrix function, $A(z)$, as the solutions of systems of linear equations, this paper gives a computational criterion for the order of a pole of $A(z)^{-1}$.

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1. Introduction

The objective of this note is to establish a computational procedure for finding the order of a pole of the inverse of an $n \times n$ meromorphic matrix function $A(z)$. This is a necessary first step in finding the coefficients in the Laurent expansion for $A^{-1}(z)$. The algorithm was first introduced by Sain and Massey [8], but only for transfer functions in realized form and hence rational matrix functions. This was extended by Howlett [7] and Avrachenkov [1] to functions $A(\varepsilon)$, which are analytic in a neighbourhood of the origin. Here, the procedure is extended to the class of meromorphic functions and a new proof is provided. The criterion simply involves the calculation of the ranks of a sequence of matrices (see Theorem 1). This can be done in a computationally efficient and reliable way, for example by using singular value decompositions (see [6]).

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To give the proof, the concept of null chains is used. Using it and a classical result of [4], the proof turns out to be straight forward. Normally, the definition for null chains of a meromorphic matrix function is given in the form of "null functions". In preparation, an equivalent definition of null chains as solutions of linear equation systems is formulated. (This approach can be found also in Section 12.4 of [2] and in [3].)

2. Preparation

Let the meromorphic matrix function $A(z)$ be analytic in a deleted neighbourhood U of z_0, and have Laurent expansion in U:

$$A(z) = \sum_{j=-v}^{\infty} (z - z_0)^j A_j, \tag{1}$$

where $A_j \in C^{n\times n}$, $A_{-v} \neq 0$. A vector-valued function $\phi(z)$, such that $\phi(z)$ is analytic at z_0, $\phi(z_0) \neq 0$, $A(z)\phi(z)$ is analytic at z_0, and $A(z)\phi(z)|_{z=z_0} = 0$, is called a *null function* of $A(z)$. The order of z_0 as a zero of $A(z)\phi(z)$ is called *the order of the null function* $\phi(z)$.

Suppose the order of a null function $\phi(z)$ is k. Develop the null function in powers of $(z - z_0)$:

$$\phi(z) = \sum_{j=0}^{\infty}(z - z_0)^j \phi_j,$$

where $\phi_j \in C^n$. It follows that $\phi_0 = \phi(z_0) \neq 0$. Vector ϕ_0 is called an *eigenvector* and $\phi_0, \phi_1, \ldots, \phi_{k-1}$ is called a *null chain* of $A(z)$ of length k corresponding to z_0. (See Sections 3.1 and 3.2 of [2]. The same definitions are given in [4], where a null function is called a root function and a null chain is called a chain of eigenvector and associated vectors.)

It will be convenient to introduce the notation

$$A^{(k)} = \begin{bmatrix} A_{-v} & 0 & 0 & \cdots & 0 \\ A_{-v+1} & A_{-v} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_k & A_{k-1} & A_{k-2} & \cdots & A_{-v} \end{bmatrix}, \tag{2}$$

a square matrix of size $n(k + v + 1)$, and also the $n(k + v + 1)$ column vector $\phi^{(v+k)}$ with entries $\phi_0, \ldots, \phi_k, \phi_{k+1}, \ldots, \phi_{k+v}$.

If $\phi_0, \phi_1, \ldots, \phi_{k-1}$ is a null chain of $A(z)$ at z_0, then

$$\begin{cases} A_{-v}\phi_0 = 0, \\ A_{-v+1}\phi_0 + A_{-v}\phi_1 = 0, \\ \cdots \\ A_{k-1}\phi_0 + \cdots + A_0\phi_{k-1} + \cdots + A_{-v}\phi_{v+k-1} = 0, \end{cases}$$

or,

$$A^{(k-1)}\phi^{(v+k-1)} = 0,$$

with $\phi_0 \neq 0$, for some $\phi_k, \ldots, \phi_{v+k-1}$. To see this, write

$$A(z)\phi(z) = \left(\sum_{j=-v}^{\infty} (z - z_0)^j A_j\right)\left(\sum_{j=0}^{\infty}(z - z_0)^j \phi_j\right),$$

and equate to zero the coefficients of $(z - z_0)^j$ for $j = -v, \ldots, 0, \ldots, k - 1$. Observe that A_{-v} is necessarily singular.

Conversely, if $\phi_0, \phi_1, \ldots, \phi_h$ is a solution of the system:

$$A^{(-v+h)}\phi^{(h)} = 0,$$

with $\phi_0 \neq 0$, $h \geqslant v$, then $\phi_0, \ldots, \phi_{h-v}$ is a null chain of $A(z)$ of length $h - v + 1$ corresponding to z_0, since using null function

$$\phi(z) = \sum_{j=0}^{h}(z - z_0)^j \phi_j + (z - z_0)^{h+1}\psi(z),$$

for some $\psi(z)$ which is analytic in a neighbourhood of z_0, it is easy to see that the order of $\phi(z)$, or the order of z_0 as the zero of $A(z)\phi(z)$, is $h - v + 1$.

Example 1. Let

$$A(z) = \begin{bmatrix} \frac{1}{z-z_0} & 0 \\ 0 & (z - z_0)^2 \end{bmatrix},$$

then

$$\phi_1(z) = \begin{bmatrix} (z - z_0)^2 \\ 1 \end{bmatrix}$$

is a null function of order 1,

$$\phi_2(z) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is a null function of order 2, so

$$\phi_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is a null chain of length 1, and

$$\phi_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \phi_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

is a null chain of length 2. Equivalently, since the system

$$\begin{cases} A_{-1}\phi_0 = 0 \\ A_0\phi_0 + A_{-1}\phi_1 = 0 \end{cases}$$

has solution

$$\phi_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \phi_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

ϕ_0 is a null chain of length 1. Since

$$\begin{cases} A_{-1}\phi_0 = 0 \\ A_0\phi_0 + A_{-1}\phi_1 = 0 \\ A_1\phi_0 + A_0\phi_1 + A_{-1}\phi_2 = 0 \end{cases}$$

has solution

$$\phi_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \phi_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \phi_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

ϕ_0, ϕ_1 is a null chain of length 2.

3. The criterion

Let

$$A^{-1}(z) = \sum_{j=-s}^{\infty} B_j(z - z_0)^j$$

with $B_{-s} \neq 0$. It is assumed that $A_{-v}, A_{-v+1}, \ldots, A_1, \ldots,$ in Eq. (1) are known. Our question is what is the order of z_0 as a pole of $A^{-1}(z)$, i.e. the number s?

By the result of Theorem 7.1 of [4], we know that s is the length of the longest null chain of $A(z)$ corresponding to z_0.

Using the notations introduced above, we can say that s is the first t such that

$$A^{(t-1)}\phi^{(t-1)} = 0 \tag{3}$$

has solution with $\phi_0 \neq 0$, and

$$A^{(t)}\phi^{(t)} = 0 \tag{4}$$

has no solution with $\phi_0 \neq 0$.

From the block structure of $A^{(t)}$ it is clear that

$$\dim\bigl(\ker A^{(t-1)}\bigr) \leqslant \dim\bigl(\ker A^{(t)}\bigr).$$

Suppose $\dim(\ker A^{(t-1)}) < \dim(\ker A^{(t)})$. Then there must be some solution $\phi^{(t)}$ of Eq. (4) such that $\phi_0 \neq 0$, meaning that a chain of length t can be continued to length $t+1$. Hence we can conclude that s is the first t such that

$$\dim\bigl(\ker A^{(t-1)}\bigr) = \dim\bigl(\ker A^{(t)}\bigr). \tag{5}$$

However

$$\dim\bigl(\ker A^{(t-1)}\bigr) = n(t+v) - \operatorname{rank} A^{(t-1)},$$

and

$$\dim\bigl(\ker A^{(t)}\bigr) = n(t+v+1) - \operatorname{rank} A^{(t)}.$$

So that Eq. (5) is equivalent to:

$$\operatorname{rank} A^{(t)} = \operatorname{rank} A^{(t-1)} + n.$$

Now we have a rank criterion for s:

Theorem 1. *Let an $n \times n$ meromorphic matrix function $A(z)$ have Laurent expansion about z_0:*

$$A(z) = \sum_{j=-v}^{\infty} (z-z_0)^j A_j, \quad \text{where } A_j \in C^{n\times n},\ A_{-v} \neq 0,$$

then the order of z_0 as a pole of $A^{-1}(z)$ is the first t such that $\operatorname{rank} A^{(t)} = \operatorname{rank} A^{(t-1)} + n$, *where $A^{(t-1)}$, $A^{(t)}$ are defined in Eq.* (2).

Note also that, because $A^{(t)}$ has the form

$$A^{(t)} = \begin{bmatrix} A^{(t-1)} & 0 \\ * & A_{-v} \end{bmatrix},$$

the ranks of $A^{(0)}, A^{(1)}, \ldots$ can be computed recursively until an increase of size n is observed.

Example 2. In Example 1,

$$A_{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

So

$$A^{(0)} = \begin{bmatrix} A_{-1} & 0 \\ A_0 & A_{-1} \end{bmatrix}$$

and $\operatorname{rank} A^{(0)} = 2$. Recursively, $\operatorname{rank} A^{(1)} = 3$, $\operatorname{rank} A^{(2)} = 5$, so 2 is the number that first makes the equation $\operatorname{rank} A^{(2)} = \operatorname{rank} A^{(2-1)} + 2$. So the order of z_0 as a pole of $A^{-1}(z)$ is 2.

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